On triangulations, quivers with potentials and mutations

Daniel Labardini-Fragoso

Abstract. In this survey article we give a brief account of constructions and results concerning the quivers with potentials associated to triangulations of surfaces with marked points. Besides the fact that the mutations of these quivers with potentials are compatible with the flips of triangulations, we mention some recent results on the representation type of Jacobian algebras. We also mention a couple of applications that the quivers with potentials associated to triangulations have had in the subject of stability conditions and in theoretical physics.

Introduction

Around 11 years ago, Fomin-Zelevinsky defined cluster algebras *in an attempt to create an algebraic framework for dual canonical bases and total positivity in semisimple groups* (cf. [**13**], from whose abstract the emphasized line was taken). Since then, cluster algebras have been found to possess interactions with a wide variety of areas, like Poisson geometry, integrable systems, Teichmüller theory, Lie theory, representation theory of associative algebras, hyperbolic 3-manifolds, commutative and non-commutative algebraic geometry, mirror symmetry, KP solitons, and even with string theory in Physics.

Fundamental in the definition of cluster algebras is the notion of *quiver mutation*, which is a combinatorial operation on quivers. In a representation-theoretic approach to cluster algebras, Derksen-Weyman-Zelevinsky developed in [**9**] a *mutation theory of quivers with potentials*, which lifts quiver mutation from the combinatorial to the algebraic level. A quiver with potential (QP for short) is a pair consisting of a quiver Q and a potential S on Q, that is, a linear combination of cycles of Q. The mutation theory of quivers with potentials ultimately leads to the notion of mutation of representations of QPs, thus providing a representation-theoretic interpretation for the combinatorial operation of quiver mutations.

On the other hand, a class of cluster algebras arising from triangulations of Riemann surfaces was introduced and systematically studied in [**11**] by Fomin-Shapiro-Thurston. These authors show that the elementary operation of *flip* of arcs in triangulations can be interpreted as the operation of mutation inside the corresponding cluster algebra. In particular, they show that every triangulation τ of a Riemann surface with marked points has a naturally associated quiver $Q(\tau)$, and that the flip of triangulations is reflected in the quiver level as quiver mutation.

In this survey article we describe a construction from [**17**] that associates a potential $S(\tau)$ to each triangulation τ, in such a way that the operation of flip

is reflected at the level of QPs as the mutation of Derksen-Weyman-Zelevinsky. Then we state some results of Geiss, Schröer and the author, on the representation type of the Jacobian algebras of the QPs $(Q(\tau), S(\tau))$. Finally, we mention a couple of applications that the QPs $(Q(\tau), S(\tau))$ have had in the subject of stability conditions and in theoretical physics.

The paper is divided in four sections. In Section 1, after recalling some elementary facts concerning (complete) path algebras of quivers (Subsection 1.1), we describe the combinatorial operation of quiver mutation (Subsection 1.2), then we give a quick overview of mutations of quivers with potentials (Subsection 1.3), and close the section with a brief reminder of the setup of surfaces with marked points, their triangulations and flips of triangulations (Subsection 1.4).

In Section 2, we quickly say how to attach a quiver $Q(\tau)$ to each triangulation τ of a surface with marked points and state the compatibility between flips and quiver mutations (Subsection 2.1). In Subsection 2.2 we lift the story to the level of QPs, that is, we describe a way to associate a QP $(Q(\tau), S(\tau))$ to each triangulation τ, and state the compatibility between flips and QP-mutations.

In Section 3 we state results regarding the finite-dimensionality and the representation type of the Jacobian algebras of the QPs $(Q(\tau), S(\tau))$. We also mention some results on the uniqueness of non-degenerate potentials for the quivers $Q(\tau)$. Finally, in Section 4 we indicate how QPs give rise to triangulated Calabi-Yau categories, and mention how, via such categories, the QPs $(Q(\tau), S(\tau))$ have had applications in the subject of stability conditions and in theoretical physics.

1. Three operations

1.1. Quivers and path algebras. Recall that a *quiver* is a finite graph with oriented edges, that is, a quadruple $Q = (Q_0, Q_1, t, h)$ consisting of a finite set of *vertices* Q_0, a finite set of *arrows*, and a pair of functions $t, h : Q_1 \to Q_0$ that determine the *tail* $t(\alpha)$ and the head $h(\alpha)$ of any given arrow $\alpha \in Q_1$. We write $\alpha : j \to i$ to indicate that $t(\alpha) = j$ and $h(\alpha) = i$. We will always assume that the quivers we work with are *loop-free*, that is, that no arrow α satisfies $h(\alpha) = t(\alpha)$.

A *path of length* $\ell > 0$ on Q is a sequence $a = \alpha_1\alpha_2 \dots \alpha_\ell$ of arrows with $t(\alpha_j) = h(\alpha_{j+1})$ for $j = 1, \dots, \ell - 1$. We set $h(a) = h(\alpha_1)$ and $t(a) = t(\alpha_\ell)$. Positive-length paths are composed as functions, that is, if $a = \alpha_1 \dots \alpha_\ell$ and $b = \beta_1 \dots \beta_{\ell'}$ are paths with $h(b) = t(a)$, then the concatenation ab is defined as the path $\alpha_1, \dots \alpha_\ell\beta_1 \dots \beta_{\ell'}$, which starts at $t(ab) = t(\beta_{\ell'})$ and ends at $h(ab) = h(\alpha_1)$.

For each vertex $i \in Q_0$ we formally introduce a *length-0 path* e_i. By A^ℓ we denote the $\mathbb{C}$-vector space with basis the set of paths of length $\ell \geq 0$. We use the notations $R = A^0$ and $A = A^1$. Note that R is the vector space with basis the set of length-0 paths, hence has dimension equal to the cardinality of Q_0, while A is the vector space with basis the set of arrows of Q. If we define $e_ie_j = \delta_{i,j}e_i$, where $\delta_{i,j}$ is the *Kronecker delta*, R becomes a commutative $\mathbb{C}$-algebra. Furthermore, if we define $e_i\alpha = \delta_{i,h(\alpha)}\alpha$ and $\alpha e_i = \delta_{i,t(\alpha)}\alpha$, then A, and actually every A^ℓ with $\ell > 0$, becomes an R-R-bimodule.

Definition 1.1. The *path algebra* of Q is the $\mathbb{C}$-vector space consisting of all finite linear combinations of paths in Q, that is,

$$R\langle Q\rangle = \bigoplus_{\ell=0}^{\infty} A^\ell.$$

The *complete path algebra* of Q is the $\mathbb{C}$-vector space consisting of all possibly infinite linear combinations of paths in Q, that is,

$$R\langle\langle Q\rangle\rangle = \prod_{\ell=0}^{\infty} A^{\ell}.$$

Both $R\langle Q\rangle$ and $R\langle\langle Q\rangle\rangle$ have their multiplications induced by concatenation of paths, so that the product of two paths is their concatenation if they can be concatenated, and 0 if they cannot be concatenated.

Note that the $R\langle Q\rangle$ is a $\mathbb{C}$-subalgebra of $R\langle\langle Q\rangle\rangle$. Actually, $R\langle Q\rangle$ is dense in $R\langle\langle Q\rangle\rangle$ under the $\mathfrak{m}$-*adic topology* of $R\langle\langle Q\rangle\rangle$. The fundamental system of open neighborhoods of this topology around 0 is given by the powers of the two-sided ideal of $R\langle\langle Q\rangle\rangle$ generated by the arrows of Q.

We are ready to describe the three operations this survey article is about: quiver mutations, mutations of quivers with potentials, and flips of surface triangulations.

1.2. Quiver mutations.

Definition 1.2. Let Q be a quiver. An *ℓ-cycle* on Q is a path $\alpha_1\alpha_2\dots\alpha_\ell$, with $\ell > 0$, such that $h(\alpha_1) = t(\alpha_\ell)$. A quiver is *2-acyclic* if it does not have 2-cycles.

Central in the definition of cluster algebras is the notion of *quiver mutation*. This is a combinatorial operation on 2-acyclic quivers that can be described as an elementary 3-step procedure as follows. Start with a 2-acyclic quiver Q and a vertex i of Q.

(Step 1) Every time we have an arrow $\alpha : j \to i$ and an arrow $\beta : i \to k$ in Q, add an arrow $[\beta\alpha] : j \to k$;
(Step 2) replace each arrow γ incident to i with an arrow γ^* going in the opposite direction;
(Step 3) delete 2-cycles one by one (2-cycles may have been created when applying Step 1).

The result is a 2-acyclic quiver $\mu_i(Q)$, called the *mutation of Q with respect to i*. See Figure 1 for an example.

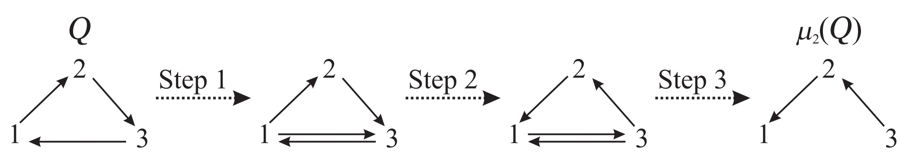

Figure 1. The three steps of quiver mutation. Here we are applying μ_2.

1.3. Mutations of quivers with potentials.

In a representation-theoretic approach to cluster algebras, Derksen-Weyman-Zelevinsky developed in [**9**] a *mutation theory of quivers with potentials*, which lifts quiver mutation from the combinatorial to the algebraic level.

Definition 1.3. Let Q be a quiver. An element S of $R\langle\langle Q\rangle\rangle$ is called a *potential* if it is a possibly infinite linear combination of cycles of Q, with the property that no two different cycles appearing in S with non-zero coefficient can be obtained

from each other by rotation. If S is a potential on Q, we say that the pair (Q,S) is a *quiver with potential*, or simply a *QP*.

Quiver mutation is lifted to the algebraic level of QPs by providing lifts of the three steps described in Subsection 1.2. Among the three steps, the one that turns out to be the hardest to lift is Step 3: one needs an algebraic procedure to delete 2-cycles algebraically. The procedure is provided by a technical result (Theorem 1.6 below) that requires some preparation.

Definition 1.4. Let Q and Q' be quivers with the same vertex set $Q_0=Q'_0$.

- Two potentials S and S' on Q are *cyclically-equivalent* if $S-S'$ lies in the closure of the vector subspace of $R\langle\langle Q\rangle\rangle$ spanned by all the elements of the form $\alpha_1\dots\alpha_\ell-\alpha_2\dots\alpha_\ell\alpha_1$ with $\alpha_1\dots\alpha_\ell$ a cycle of positive length.
- We say that two QPs (Q,S) and (Q',S') are *right-equivalent* if there exists a *right-equivalence* between them, that is, a ka-algebra isomorphism $\varphi:R\langle\langle Q\rangle\rangle\to R\langle\langle Q'\rangle\rangle$ satisfying $\varphi(e_i)=e_i$ for all $i\in Q_0=Q'_0$, and such that $\varphi(S)$ is cyclically-equivalent to S'.
- For each arrow $\alpha\in Q_1$ and each cyccle $\alpha_1\dots\alpha_\ell$ in Q we define the *cyclic derivative*
$$\partial_\alpha(\alpha_1\dots\alpha_\ell)=\sum_{k=1}^{\ell}\delta_{\alpha,\alpha_k}\alpha_{k+1}\dots\alpha_\ell\alpha_1\dots\alpha_{k-1},$$
and extend ∂_α by linearity and continuity so that $\partial_\alpha(S)$ is defined for every potential S.
- The *Jacobian ideal* $J(S)$ is the topological closure of the two-sided ideal of $R\langle\langle Q\rangle\rangle$ generated by $\{\partial_\alpha(S)\mid\alpha\in Q_1\}$, and the *Jacobian algebra* $\mathcal{P}(Q,S)$ is the quotient algebra $R\langle\langle Q\rangle\rangle/J(S)$.
- A QP (Q,S) is *trivial* if $S\in A^2$ and $\{\partial_\alpha(S)\mid\alpha\in Q_1\}$ spans A as a $\mathbb{C}$-vector space.
- A QP (Q,S) is *reduced* if the degree-2 component of S is 0, that is, if the expression of S involves no 2-cycles.
- The *direct sum* $Q\oplus Q'$ is the quiver whose vertex set is $Q_0=Q'_0$ and whose arrow set is the disjoint union $Q_1\sqcup Q'_1$, with the tail and head functions defined in the obvious way.
- The *direct sum* of two QPs (Q,S) and (Q',S') is the QP $(Q,S)\oplus(Q',S')=(Q\oplus Q',S+S')$.

Proposition 1.5. [**9**] *If $\varphi:R\langle\langle Q\rangle\rangle\to R\langle\langle Q'\rangle\rangle$ is a right-equivalence between (Q,S) and (Q',S'), then φ sends $J(S)$ onto $J(S')$ and therefore induces an algebra isomorphism $\mathcal{P}(Q,S)\to\mathcal{P}(Q',S')$.*

Theorem 1.6. [**9**] *For every QP (Q,S) there exist a trivial QP $(Q_{\mathrm{triv}},S_{\mathrm{triv}})$ and a reduced QP $(Q_{\mathrm{red}},S_{\mathrm{red}})$ such that (Q,S) is right-equivalent to the direct sum $(Q_{\mathrm{triv}},S_{\mathrm{triv}})\oplus(Q_{\mathrm{red}},S_{\mathrm{red}})$. The right-equivalence class of each of the QPs $(Q_{\mathrm{triv}},S_{\mathrm{triv}})$ and $(Q_{\mathrm{red}},S_{\mathrm{red}})$ is determined by the right-equivalence class of (Q,S).*

In the situation of Theorem 1.6, the QPs $(Q_{\mathrm{red}},S_{\mathrm{red}})$ and $(Q_{\mathrm{triv}},S_{\mathrm{triv}})$ are called, respectively, the *reduced part* and the *trivial part* of (Q,S).

Remark 1.7. Theorem 1.6 is the reason why we work with the complete path algebra $R\langle\langle Q\rangle\rangle$ rather than with the path algebra $R\langle Q\rangle$: The proof of Theorem 1.6

involves a limit process; the only way to ensure convergence of this limit process is to work with the complete path algebra.

We now turn to the definition of mutation of a QP. Let (Q, S) be a QP on the vertex set Q_0 and let $i \in Q_0$. Assume that Q is 2-acyclic. Let $\widetilde{\mu}_i(Q)$ be the quiver obtained right after applying the first two steps of quiver mutation, but before applying Step 3. Replacing S if necessary with a cyclically equivalent potential, we assume that every cycle appearing in the expression of S starts at a vertex different from i. Then we define $[S]$ to be the potential on $\widetilde{\mu}_i(Q)$ obtained from S by replacing each length-2 path $\beta\alpha$ of Q such that $h(\alpha) = i = t(\beta)$, with the arrow $[\beta\alpha]$ of $\widetilde{\mu}_i(Q)$. Also, we define $\Delta_i(Q) = \sum \alpha^*\beta^*[\beta\alpha]$, where the sum runs over all length-2 paths $\beta\alpha$ of Q such that $h(\alpha) = i = t(\beta)$. Finally, we set $\widetilde{\mu}_i(S) = [S] + \Delta_i(Q)$, which clearly is a potential on $\widetilde{\mu}_i(Q)$.

Definition 1.8. [**9**] Under the assumptions and notation just stated, we define the *mutation* $\mu_i(Q, S)$ of (Q, S) with respect to i to be the reduced part of the QP $\widetilde{\mu}_i(Q, S) = (\widetilde{\mu}_i(Q), \widetilde{\mu}_i(S))$.

"Unfortunately", given a QP (Q, S) with Q 2-acyclic, the underlying quiver of the mutated QP $\mu_i(Q, S)$ is not necessarily 2-acyclic, its 2-acyclicity depends heavily on the potential S.

Definition 1.9. [**9**] A QP (Q, S) is *non-degenerate* if Q is 2-acyclic and the underlying quiver of the QP obtained after any possible sequence of QP-mutations is 2-acyclic.

Theorem 1.10. [**9**]

(1) *Mutations of QPs are well defined up to right-equivalence.*
(2) *Mutations of QPs are involutive up to right-equivalence.*
(3) *Every 2-acyclic quiver Q admits a non-degenerate potential on it.*
(4) *Finite-dimensionality of Jacobian algebras is invariant under QP-mutations.*

1.4. Flips of triangulations.

Definition 1.11. A *surface with marked points*, or simply a *surface*, is a pair $(\Sigma, \mathbb{M})$, where Σ is a compact connected oriented Riemann surface with (possibly empty) boundary, and $\mathbb{M}$ is a non-empty finite subset of Σ containing at least one point from each connected component of the boundary of Σ. We refer to the elements of $\mathbb{M}$ as *marked points*. The marked points that lie in the interior of Σ are called *punctures*, and the set of punctures of $(\Sigma, \mathbb{M})$ is denoted $\mathbb{P}$.

We think of $\mathbb{M}$ as a prescribed set of vertices for triangulations of Σ. More formally:

Definition 1.12. Let $(\Sigma, \mathbb{M})$ be a surface with marked points.

(1) An *arc* on $(\Sigma, \mathbb{M})$, is a curve i on Σ such that:
   - the endpoints of i belong to $\mathbb{M}$;
   - i does not intersect itself, except that its endpoints may coincide;
   - the points in i that are not endpoints do not belong to $\mathbb{M}$ nor to the boundary of Σ;
   - i does not cut out an unpunctured monogon nor an unpunctured digon.

(2) Two arcs i_1 and i_2 are *isotopic* rel $\mathbb{M}$ if there exists an isotopy $H : I\times\Sigma \to \Sigma$ such that $H(0,x) = x$ for all $x \in \Sigma$, $H(1,i_1) = i_2$, and $H(t,m) = m$ for all $t \in I$ and all $m \in \mathbb{M}$). Arcs will be considered up to isotopy rel $\mathbb{M}$, parametrization, and orientation.
(3) Two arcs are *compatible* if there are arcs in their respective isotopy classes that, except possibly for their endpoints, do not intersect.
(4) An *ideal triangulation* of $(\Sigma,\mathbb{M})$ is any maximal collection τ of pairwise compatible arcs.

REMARK 1.13. The term *ideal triangulation* comes from the connection with Teichmüller theory, see [**12**].

Any ideal triangulation τ of $(\Sigma,\mathbb{M})$ splits Σ into *triangles*. If $(\Sigma,\mathbb{M})$ has punctures, some of the triangles of τ may be *self-folded*, see Figure 2. A self-folded triangle always contains a *folded side*. If $i \in \tau$ is an arc which is not the folded side

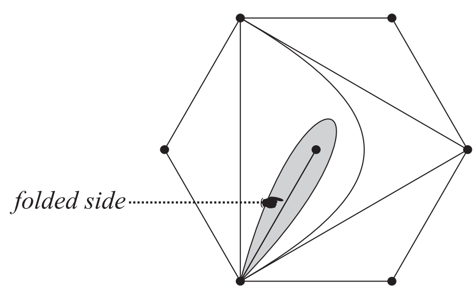

FIGURE 2. Ideal triangulation with a self-folded triangle (the self-folded triangle has been highlighted).

of a self-folded triangle, then there exists a unique arc $j \neq i$ on $(\Sigma,\mathbb{M})$, such that $\sigma = (\tau \setminus \{i\}) \cup \{j\}$ is an ideal triangulation of $(\Sigma,\mathbb{M})$. We say that σ is obtained from τ by the *flip* of the arc i. See Figure 3. In order to be able to flip folded

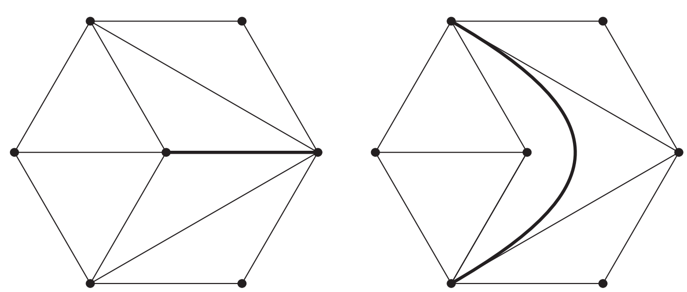

FIGURE 3. Two ideal triangulations related by a flip, the arcs involved in the flip have been drawn bolder than the rest.

sides of self-folded triangles, Fomin-Shapiro-Thurston introduced in [**11**] the concept of *tagged triangulation*, a notion of triangulation which is more general than the notion of ideal triangulation we have defined above. As the reader may already expect, the combinatorics of flips of tagged triangulations becomes rather subtle.

2. The quiver with potential of a triangulation

2.1. The quiver of a triangulation. Every ideal triangulation τ has a quiver $Q(\tau)$ associated in a natural way. This was first observed by Fock-Goncharov [**10**], Fomin-Shapiro-Thurston [**11**] and Gekhtman-Shapiro-Vainshtein [**15**]. Let us describe $Q(\tau)$ under the assumption that

(2.1) every puncture of $(\Sigma, \mathbb{M})$ is incident to at least three arcs of τ.

The vertices of $Q(\tau)$ are the arcs of τ, the arrows are drawn in the clockwise direction within each triangle of τ. See Figure 4.

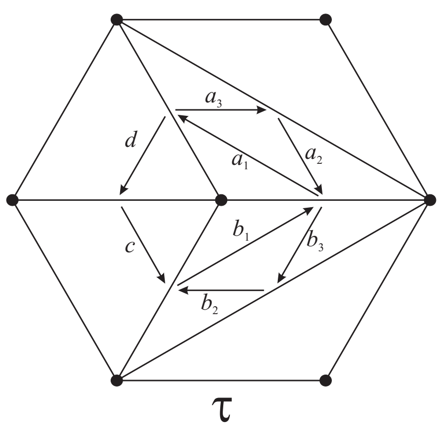

FIGURE 4. Two ideal triangulations related by a flip, with their associated quivers drawn on the surface.

If an ideal triangulation τ is such that

(2.2) there are punctures incident to less than three arcs of τ,

the definition of $Q(\tau)$ is slightly more involved, but we stress that all triangulations, including the tagged ones, have naturally associated quivers. Let us also remark that the definition of the quivers of tagged triangulations is due to Fomin-Shapiro-Thurston alone.

THEOREM 2.1. [**10, 11, 15**] *Let τ and σ be ideal triangulations of $(\Sigma, \mathbb{M})$. If σ is obtained from τ by the flip of an arc i, then $Q(\sigma) = \mu_i(Q(\tau))$. That is, if two ideal triangulations are related by a flip, then their associated quivers are related by the corresponding quiver mutation.*

Thus, for example, the two quivers drawn in Figure 4 are related by quiver mutation. Fomin-Shapiro-Thurston have shown that Theorem 2.1 is valid in the more general setting of tagged triangulations.

2.2. The potential of a triangulation. We know that every triangulation has a quiver associated to it, and we know that flips of triangulations are compatible with mutations of quivers. Could this story be lifted to the level of QPs? To try and answer this question, we first notice that for each ideal triangulation τ of $(\Sigma, \mathbb{M})$ satisfying (2.1), the quiver $Q(\tau)$ possesses two obvious types of cycles:

- 3-cycles arising from triangles $\triangle$ of τ, and

- simple cycles (that is, without repeated arrows) surrounding the punctures.

To avoid redundancies, in the following definition we consider cycles up to cyclical equivalence. That is, whenever a cycle ξ is considered, the cycles cyclically-equivalent to ξ, but different from ξ, are ignored.

Definition 2.2. [**17**] Let τ be a triangulation of $(\Sigma, \mathbb{M})$ satisfying (2.1). The potential $S(\tau)$ associated to τ is the potential on $Q(\tau)$ that results from adding all the 3-cycles that arise from triangles of τ and all the simple cycles that surround the punctures of $(\Sigma, \mathbb{M})$.

Remark 2.3.
- In the particular case when $(\Sigma, \mathbb{M})$ is a surface without punctures and non-empty boundary, the potentials $S(\tau)$ were found independently by Assem-Brüstle-Charbonneau-Plamondon in [**5**].
- In [**17**], the definition of $S(\tau)$ was given for every ideal triangulation τ, including those satisfying (2.2).

Example 2.4. The potentials associated to the ideal triangulations τ and σ shown in Figure 4 are $S(\tau) = a_1a_2a_3 + b_1b_2b_3 + a_1b_1cd$ and $S(\sigma) = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 + \alpha_1cd$. The QPs $(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ turn out to be related by QP-mutation. This can be checked directly, or seen as a consequence of the following theorem.

Theorem 2.5. [**17**] *Let τ and σ be ideal triangulations of a surface with marked points $(\Sigma, \mathbb{M})$. If τ and σ are related by the flip of an arc i, then $(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ are related by the QP-mutation μ_i. More precisely, the QPs $\mu_i(Q(\tau), S(\tau))$ and $(Q(\tau), S(\tau))$ are right-equivalent. If, moreover, the boundary of Σ is not empty, then all QPs $(Q(\tau), S(\tau))$ associated to the ideal triangulations of $(\Sigma, \mathbb{M})$ are non-degenerate.*

In the more general situation of tagged triangulations, potentials were defined in [**8**] under the assumption that the underlying surface Σ has non-empty boundary, but even with this assumption, the corresponding "tagged version" of Theorem 2.5 was not proved for all flips of tagged triangulations. The referred empty-boundary assumption was later removed in [**18**].

Theorem 2.6. [**18**] *Let $(\Sigma, \mathbb{M})$ be a surface with marked points. Suppose $(\Sigma, \mathbb{M})$ is not one of the following:*
- *a sphere[1] with less than 9 punctures;*
- *a positive-genus surface with empty boundary and exactly 2, 3, 4 or 5 punctures.*

If τ and σ are tagged triangulations of $(\Sigma, \mathbb{M})$ related by the flip of a tagged arc i, then $(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ are related by the QP-mutation μ_i. Consequently, all QPs $(Q(\tau), S(\tau))$ associated to the ideal triangulations of $(\Sigma, \mathbb{M})$ are non-degenerate.

3. On the Jacobian algebras

From the perspective of representation theory of associative algebras, there are several natural questions one can ask regarding the Jacobian algebras of the QPs

[1]that is, a genus-0 surface with empty boundary

$(Q(\tau), S(\tau))$. For example: Are they finite-dimensional?, are they tame/wild? And from the cluster algebra point of view, one can ask whether the quiver $Q(\tau)$ admits more than one non-degenerate potential up to right-equivalence.

THEOREM 3.1. [**17**] *Let $(\Sigma, \mathbb{M})$ be a surface with non-empty boundary (and any number of punctures). Then for any ideal triangulation τ of $(\Sigma, \mathbb{M})$, the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$ has finite dimension over $\mathbb{C}$.*

THEOREM 3.2. [**19**] *Let $(\Sigma, \mathbb{M})$ be a surface with empty boundary. Suppose that $(\Sigma, \mathbb{M})$ is not a sphere with less than 5 punctures. Then for any ideal triangulation τ of $(\Sigma, \mathbb{M})$, the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$ has finite dimension over $\mathbb{C}$.*

REMARK 3.3.
- Theorem 3.1 can be either proved independently of Theorem 3.2, or deduced from it via *restriction of QPs*.
- In the particular case of unpunctured surfaces with non-empty boundary, Theorem 3.1 was proved by Assem-Brüstle-Charbonneau-Plamondon in [**5**] independently of [**17**] and [**19**].
- Theorem 3.2 is due to Ladkani. In the particular case of spheres with at least five punctures, it was proved independently by Trepode–Valdivieso-Díaz in [**20**].

Let us turn to the problem of whether the Jacobian algebras $\mathcal{P}(Q(\tau), S(\tau))$ are tame or wild. In the particular case when $(\Sigma, \mathbb{M})$ is a surface with non-empty boundary and without punctures, Assem-Brüstle-Charbonneau-Plamondon have shown in [**5**] that $\mathcal{P}(Q(\tau), S(\tau))$ is a *gentle algebra*, and this implies its tameness, for gentle algebras are well-known to be tame. More generally, we have:

THEOREM 3.4. [**14**]
- *For any QP (Q, S), if $\mathcal{P}(Q, S)$ is tame, then $\mathcal{P}((\mu_i(Q, S))$ is tame as well.*
- *If $(\Sigma, \mathbb{M})$ is a surface with non-empty boundary (and any number of punctures), then there exists an ideal triangulation τ of $(\Sigma, \mathbb{M})$ such that the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$ is* clannish*, hence tame.*
- *If $(\Sigma, \mathbb{M})$ is a surface with empty boundary, then there exists an ideal triangulation τ of $(\Sigma, \mathbb{M})$ such that the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$ degenerates to a gentle algebra and hence is tame.*
- *Consequently, for every surface $(\Sigma, \mathbb{M})$ and every tagged triangulation τ of $(\Sigma, \mathbb{M})$, the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau))$ is tame.*

Concerning the question of uniqueness of non-degenerate potentials, we have:

THEOREM 3.5. [**14**] *Let $(\Sigma, \mathbb{M})$ be a surface with non-empty boundary and τ be any tagged triangulation of $(\Sigma, \mathbb{M})$. Then, up to right-equivalence, $S(\tau)$ is the only non-degenerate potential on $Q(\tau)$ provided $(\Sigma, \mathbb{M})$ is not an unpunctured torus with exactly one boundary component and exactly one marked point on such component. For the latter surface, $Q(\tau)$ admits exactly two non-degenerate potentials up to right-equivalence.*

THEOREM 3.6. [**14**] *Let $(\Sigma, \mathbb{M})$ be a torus with exactly one marked point, with the boundary of Σ either empty or non-empty. Let τ be a tagged triangulation of $(\Sigma, \mathbb{M})$. Then $Q(\tau)$ admits a non-degenerate potential W whose Jacobian algebra $\mathcal{P}(Q(\tau), W)$ is wild. The potential W is not right-equivalent to $S(\tau)$.*

4. Some applications

Quivers with potentials are a tool to obtain categories. Besides the module categories of Jacobian algebras, there are other categories associated to QPs that are of interest not only to representation-theorists, but to authors from other areas as well. We give a very rough description of a couple of these categories, the reader is referred to [**3**] and [**4**, Section 3] for precise definitions, statements and citations. Let (Q,S) be a non-degenerate QP with finite-dimensional Jacobian algebra. One defines the *complete Ginzburg dg algebra*[2] as certain $\mathbb{Z}$-graded algebra $\widehat{\Gamma}(Q,S)$ that has the complete path algebra $R\langle\langle Q\rangle\rangle$ sitting as its degree-0 component. The algebra $\widehat{\Gamma}(Q,S)$ has a differential induced by the cyclic derivatives of S and the so-called *relations of the preprojective algebra*. Via the *derived category* of the category of dg modules over $\widehat{\Gamma}(Q,S)$, one arrives at a *3-Calabi-Yau triangulated* category $\mathcal{D}\widehat{\Gamma}(Q,S)$. Amiot then defines the *generalized cluster category* $\mathcal{C}(Q,S)$ as certain quotient of a subcategory of $\mathcal{D}\widehat{\Gamma}(Q,S)$. The generalized cluster category turns out to be Hom-*finite 2-Calabi-Yau triangulated*. Amiot and Keller-Yang show that QP-mutations induce equivalences of categories:

THEOREM 4.1. [**16, 3, 4**] *The categories $\mathcal{D}\widehat{\Gamma}(Q,S)$ and $\mathcal{C}(Q,S)$ are respectively equivalent to $\mathcal{D}\widehat{\Gamma}(\mu_i(Q,S))$ and $\mathcal{C}(\mu_i(Q,S))$ as triangulated categories.*

A combination of Theorems 2.6 and 4.1 yields:

THEOREM 4.2. *Every surface $(\Sigma,\mathbb{M})$ from Theorem 2.6 gives rise to:*

(1) *a 3-Calabi-Yau triangulated category $\mathcal{D}(\Sigma,\mathbb{M})$, with canonical hearts and tilts of canonical hearts combinatorially interpreted as tagged triangulations and flips of tagged triangulations, respectively.*
(2) *A* Hom-*finite 2-Calabi-Yau triangulated category $\mathcal{C}(\Sigma,\mathbb{M})$, with reachable cluster-tilting objects and IY-mutations of reachable cluster-tilting objects combinatorially interpreted as tagged triangulations and flips of tagged triangulations, respectively*[3]*; cf.* [**4**, Section 3.4], [**8**, Theorem 4.10].

Indeed, one defines $\mathcal{D}(\Sigma,\mathbb{M}) = \mathcal{D}\widehat{\Gamma}(Q(\tau),S(\tau))$ and $\mathcal{C}(\Sigma,\mathbb{M}) = \mathcal{C}(Q(\tau),S(\tau))$ for any tagged triangulation τ of $(\Sigma,\mathbb{M})$. These categories are independent of τ by Theorems 2.6 and 4.1.

Ongoing work [**6**] of Bridgeland-Smith shows that spaces of *Bridgeland stability conditions* on the categories $\mathcal{D}(\Sigma,\mathbb{M})$ can be realized as spaces of *quadratic differentials* on the Riemann surface Σ. When Σ has empty boundary, they furthermore show that $\mathcal{D}(\Sigma,\mathbb{M})$ can be interpreted as the Fukaya category of a symplectic 6-manifold underlying certain Calabi-Yau-3 varieties that fiber over the surface Σ.

In physics, the QPs $(Q(\tau),S(\tau))$, as well as their QP-mutation compatibility with flips, have been used by Alim-Cecotti-Cordova-Espahbodi-Rastogi-Vafa [**1**], [**2**], and Cecotti [**7**], in their study of $N=2$ *quantum field theories* and associated *BPS quivers and spectra*.

---

[2] *dg* for *differential graded*

[3] When $(\Sigma,\mathbb{M})$ is a once-punctured surface with empty boundary, this statement needs a slight refinement.

Acknowledgements

The present survey article is mainly based on a talk contributed by the author to the *Taller de Vinculación Matemáticos Mexicanos Jóvenes en el Mundo*, held at the *Centro de Investigación en Matemáticas* (CIMAT), Guanajuato, México, in August 2012. I am grateful to the organizers of this meeting, Noé Bárcenas-Torres, Fernando Galaz-García and Mónica Moreno-Rocha, for the opportunity of presenting my work.

References

[1] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, C. Vafa. *BPS Quivers and Spectra of Complete N=2 Quantum Field Theories*. arXiv:1109.4941

[2] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, C. Vafa. *N=2 Quantum Field Theories and Their BPS Quivers*. arXiv:1112.3984

[3] C. Amiot. *Cluster categories for algebras of global dimension 2 and quivers with potential*. Annales de l'Institut Fourier, 59 no. 6 (2009), 2525-2590. arXiv:0805.1035

[4] C. Amiot. *On generalized cluster categories*. ICRA XIV Proceedings, Eur. Math. Soc. (2011). arXiv:1101.3675

[5] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, P-G. Plamondon. *Gentle algebras arising from surface triangulations*. Algebra & Number Theory Vol. 4 (2010), No. 2, 201-229. arXiv:0903.3347

[6] T. Bridgeland, I. Smith. *Quadratic differentials as stability conditions*. Preprint. Private communication.

[7] S. Cecotti. *Categorical Tinkertoys for N=2 Gauge Theories*. arXiv:1203.6734

[8] G. Cerulli Irelli, D. Labardini-Fragoso. *Quivers with potentials associated to triangulated surfaces, part III: Tagged triangulations and cluster monomials*. Compositio Mathematica **148** (2012), No. 06, 1833–1866. arXiv:1108.1774

[9] H. Derksen, J. Weyman, A. Zelevinsky. *Quivers with potentials and their representations I: Mutations*. Selecta Math. 14 (2008), no. 1, 59119. arXiv:0704.0649

[10] V. V. Fock, A. B. Goncharov, *Dual Teichmüller and lamination spaces*. Handbook of Teichmüller theory, vol. I, 647-684, IRMA Lectures in Mathematics and Theoretical Physics 11, Eur. Math. Soc., 2007. arXiv:math.DG/0510312

[11] S. Fomin, M. Shapiro, D. Thurston. *Cluster algebras and triangulated surfaces, part I: Cluster complexes*. Acta Mathematica 201 (2008), 83-146. arXiv:math.RA/0608367

[12] S. Fomin, D. Thurston. *Cluster algebras and triangulated surfaces. Part II: Lambda lengths*. arXiv:1210.5569

[13] S. Fomin, A. Zelevinsky. *Cluster algebras I: Foundations*. J. Amer. Math. Soc. 15 (2002), no. 2, 497–529. arXiv:math/0104151

[14] C. Geiss, D. Labardini-Fragoso, J. Schröer. Preprint in preparation.

[15] M. Gekhtman, M. Shapiro, A. Vainshtein. *Cluster algebras and Weil-Petersson forms*. Duke Math. J. 127 (2005) 291311, math.QA/0309138

[16] B. Keller, D. Yang. *Derived equivalences from mutations of quivers with potential*. Advances in Mathematics 226 (2011), 2118–2168.

[17] D. Labardini-Fragoso. *Quivers with potentials associated to triangulated surfaces*. Proc. London Mathematical Society (2009) 98 (3): 797-839. arXiv:0803.1328

[18] D. Labardini-Fragoso. *Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions*. arXiv:1206.1798

[19] S. Ladkani. *On Jacobian algebras from closed surfaces*. arXiv:1207.3778

[20] S. Trepode, Y. Valdivieso-Díaz. *On finite dimensional Jacobian Algebras*. arXiv:1207.1917

Mathematisches Institut, Universität Bonn, Germany
E-mail address: `labardini@math.uni-bonn.de`